Application of Singular Value Decomposition Technique to System Identification by Doping an Optimum Signal

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ABSTRACT

This paper proposes a system identification methodology by using eigensystem realization algorithm (ERA) with a doping signal such that noise effect can be attenuated. The doping signal is obtained by integrating optimization technique with singular value decomposition (SVD) technique. Theoretical analysis is performed to interpret the necessity of using the optimum signal and to explain the relationship between SVD and optimum signal. Simulation results show that the proposed ERA with a novel optimum signal can achieve higher accurate results as compared to that without doping the optimum signal.

INTRODUCTION

In the past decades, there are many techniques developed for system identification process. One of the state space approaches is the ERA method developed by Juang [1,2] where the pulse response is applied to identify the state space model. The ERA approach can obtain the system matrices (A,B,C,D) for a MIMO system and no predefined order is needed.

The other state space approach is the subspace method [3-6]. This approach utilizes the extended observability matrix of a system to estimate state space representation of the system. Although the identification results of these methods are acceptable for low frequency modes, the results become worse for high frequency modes because of noise effects. A common approach is to apply a low pass filter such as Butterworth filter to remove noise, but the filter might change the phase of signal and increase the order of the system by imposing the filter dynamics on the original signal. Moreover, the ERA approach can perform the filtering behavior by truncating some of the singular vectors, applying low pass filter on pulse responses makes only little difference for identification of high order mode based on the simulation which is demonstrated later.

Many researchers have applied the SVD technique on noise filtering; for example, Broomhead[7] applied the SVD to extract qualitative dynamics from experimental data. Navarro-Esbri[8] also applied SVD and proposed a method to use the data itself to estimate the noise level then remove the noise accurately. Shin[9] used the algorithm of reconstruction of phase portrait in [7]. He changed the sampling rate and embedding dimension to reconstruct the signal from noisy data. Liu[10] also used the algorithm in [7] to filter the noise for modal parameter estimation, and the relation between singular value distribution and S/N ratio has been studied. Allen[11] proposed a generalization of singular spectrum analysis to filter the colored noise by a coordinate transformation to make noise has equal variance in all directions then separate the noise with the desired signal. Liu[12] integrated the least square method with the SVD to improve the instability of dynamic force identification. Tikhonov filter was applied as a weight of different singular vectors and integrated with the least square method to determine the parameter of Tikhonove filter. Qin[13]
utilized the cross-correlation matrices of random decrement function (RDF) to replace normal RDF to build Hankel matrices. It is shown that a cross-correlation matrix of RDF can filter noisy data by taking average on the summation of signals for different phase shift.

This paper performs some fundamental studies on the SVD filtering techniques first. Based on the analyzed results, a novel concept called optimal doping signal SVD technique is proposed. By integrating this technique with the ERA, one can achieve better results than the conventional filter approach.

SYSTEM IDENTIFICATION BY EIGENSYSTEM REALIZATION ALGORITHM (ERA)

The procedure of the ERA is described as the follows:
Assume the system is described by the discrete-time state space representation:

\[
\begin{align*}
q(k+1) &= Aq(k) + Bu(k) \\
w(k) &= Cq(k) + Du(k),
\end{align*}
\]

where, \(q(k)\) is the state vector, \(w(k)\) is output vector, \(u(k)\) is an input vector and \(A, B, C, D\) is the discrete system matrices. To obtain the system matrix \((A, B, C, D)\), the Hankel matrix \(\hat{H}(0)\) and \(\hat{H}(1)\) should be formed first, and are given as:

\[
\hat{H}(0) = \begin{bmatrix}
p(1) & p(2) & \cdots & p(N+1) \\
p(2) & p(3) & \cdots & p(N+2) \\
\vdots & \vdots & \ddots & \vdots \\
p(N+1) & p(N+2) & \cdots & p(2N+2)
\end{bmatrix},
\]

\[
\hat{H}(1) = \begin{bmatrix}
p(2) & p(3) & \cdots & p(N+2) \\
p(3) & p(4) & \cdots & p(N+3) \\
\vdots & \vdots & \ddots & \vdots \\
p(N+2) & p(N+3) & \cdots & p(2N+2)
\end{bmatrix},
\]

where \(p(i)\) is the pulse responses at time \(i\) and excited by the pulse forces \(u(k)\), \(2N+2\) is the number of sampling points.

By formulating the Hankel matrix \(\hat{H}(0)\), \(\hat{H}(1)\), and further using the singular value decomposition on \(\hat{H}(0)\), the matrix \(\hat{H}(0)\) can be represented by \(R, \Sigma, \hat{S}_n\), \(R, \Sigma, \hat{S}_n\) are orthonormal set, and \(\Sigma_n\) is a diagonal matrix given as \(\Sigma_n = \text{diag}[^{\sigma_1, \sigma_2, \ldots, \sigma_n}]\).

\(\sigma_1 \sim \sigma_n\) are the singular values of \(\hat{H}(0)\). Assuming that \(\hat{A}, \hat{B}, \hat{C}, \hat{D}\) is an estimated system model, the matrices \(\hat{A}, \hat{B}, \hat{C}, \hat{D}\) can then be obtained as:

\[
\hat{A} = \Sigma_n^{\frac{1}{2}} R_n^{\dagger} \hat{H}(1) \Sigma_n^{\frac{1}{2}}, \quad \hat{B} = \Sigma_n^{\frac{1}{2}} \hat{S}_n^T E_r, \quad (4)
\]

\[
\hat{C} = E_n^{\dagger} R_n \Sigma_n^{\frac{1}{2}}, \quad \hat{D} = p(0)
\]

where \(E_n = [I_n, O_n, \ldots, O_n]\), \(m\) is the number of outputs, \(E_r = [I_r, O_n, \ldots, O_n]\), \(r\) is the number of inputs, and \(n\) is the system order. Other \(N-n\) items have been omitted from \(\Sigma_n, R_n, \hat{S}_n\) to determine proper order of system and filter the noise.

SINGULAR VALUE DECOMPOSITION (SVD) FILTERING TECHNIQUES

To implement the ERA technique, it is important that the pulse response signal \(p(k)\) should be filtered as clean as possible. Otherwise, the formed Hankel matrix will be deteriorated by noise, and the identified results might be inaccurate, especially for high frequency mode dynamics.

The filtering behavior of the SVD can be explained in the following:

For illustration purpose, a second order system including output noise is assumed, and the corresponding pulse response function \(\hat{p}(k)\) is composed of two signals \(p_n(k)\) and \(p_o(k)\), \(p_n(k)\) is the desired signal (pulse response), and \(p_o(k)\) is the noise signal. Therefore,

\[
\hat{p}(k) = p_n(k) + p_o(k). \quad (5)
\]

The singular value decomposition of \(\hat{H}(0)\) formed by \(\hat{p}(k)\) is given as:

\[
\hat{H}(0) = \hat{R}_n \Sigma_n \hat{S}_n^T, \quad (6)
\]

where

\[
\hat{R}_n = \begin{bmatrix} \hat{f}_1 & \hat{f}_2 & \ldots & \hat{f}_n \end{bmatrix}, \quad \Sigma_n = \begin{bmatrix} \hat{\sigma}_1 & 0 & \cdots & \cdots \\ 0 & \hat{\sigma}_2 & \ddots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \hat{\sigma}_n \end{bmatrix}, \quad \hat{S}_n = \begin{bmatrix} \hat{s}_1^T \\ \hat{s}_2^T \\ \vdots \\ \hat{s}_n^T \end{bmatrix}
\]
\( \hat{r}_i, \hat{s}_i \) are the singular vectors of \( \hat{H}(0) \) with dimension \( N \times 1 \) and \( \| \hat{r}_i \| = \| \hat{s}_i \| = 1 \). \( \hat{r}_i, \hat{s}_i \) are two orthonormal sets, that is \( \hat{r}_i^T \hat{r}_j = \hat{s}_i^T \hat{s}_j = 0 \) for \( i \neq j \).

When S/N is large enough, \( \sigma_1, \sigma_2, s_1, s_2 \) are normally decomposed from the signal \( p_j(k) \) and \( \sigma_3 - \sigma_N s_3 - s_N \) are obtained from the \( p_n(k) \).

The number of the nonzero singular values is \( n \) (\( N \geq n \)). \( n \) normally represents the necessary information of a signal. For example, if \( \hat{p}(k) \) is a clean single frequency sinusoidal wave, then \( n \) is equal to 2 because a sinusoidal signal contains magnitude and phase information. The shape and frequency information is contained in singular vectors. That is the singular vector resembles the sinusoidal wave but with a phase shift.

Observe from Eq. (2) where the signal \( \hat{p}(k) \) is the first row of \( \hat{H}(0) \). Therefore, from Eq. (6), the \( \hat{p}(k), k = 1 - N + 1 \) can be expressed as:

\[
\hat{p}(k) = \hat{r}_1 \hat{\sigma}_1 \hat{s}_1^T + \hat{r}_2 \hat{\sigma}_2 \hat{s}_2^T + \ldots \hat{r}_n \hat{\sigma}_n \hat{s}_n^T,
\]

(7).

If \( \sigma_1 - \sigma_N s_1 - s_N \) are completely related to the noise signal, then the desired signal can be obtained from the following equation:

\[
\hat{p}_i(k) = \hat{\sigma}_1 \hat{s}_1^T \hat{r}_i + \hat{\sigma}_2 \hat{s}_2^T \hat{r}_i,
\]

(8)

where \( \hat{r}_i \) is the first value of the vector \( \hat{r}_i \). However, the assumption might not be valid because noise signal might pollute the process and deteriorate the performance.

Now if one applies the SVD on the pure \( p_j(k) \) in Eq. (5) without including the noise signal \( p_n(k) \), then \( p_j(k) \) can be expressed as:

\[
p_j(k) = r_1 \sigma_1 s_1^T + r_2 \sigma_2 s_2^T,
\]

(9)

where \( s_1 \) and \( s_2 \) are the singular vectors of \( H_j(0) \) formed by \( p_j(k) \), \( r_1 \) is the first value of \( r_1 \). As compared to (8) and (9), usually \( s_i \neq \hat{s}_i \) for \( i = 1, 2 \) because of the appearance of the noise.

To illustrate this phenomenon, Figure 1 shows an example when \( p_j(k) \) is a sinusoid signal and \( p_n(k) \) is zero.

![Fig. 1. SVD decomposition of a sinusoid signal under no polluted noise polluted](image1)

then SVD can decompose \( \hat{p}(k) \) very well, and \( \hat{s}_1, \hat{s}_2 \) only contain the information of \( p_j(k) \). But if \( p_n(k) \) is not zero as shown in Fig. 2., then SVD cannot decompose \( \hat{p}(k) \) very well and \( \hat{s}_i \neq \hat{s}_i \) for \( i = 1, 2 \). Here \( p_n(k) \) is assumed to be a high frequency pure sinusoidal noise. Figure 2 also shows that \( \hat{s}_1, \hat{s}_2 \) contain the information of \( p_j(k) \) and \( p_n(k) \).

![Fig. 2. SVD decomposition of a sinusoid signal well under polluted noise condition](image2)

It is expected that an ideal decomposition of SVD can decompose the signal \( \hat{p}(k) \) in Eq. (7) to let the following become true:

\[
\hat{s}_i(k) = s_i(k), \quad i = 1, 2,
\]

(10)

If Eq. (10) is valid, then the signal \( p_j(k) \) can easily be reconstructed by Eq. (8). It means the noise can be perfectly removed by the SVD. However, due to the
The decomposition can be expected. The conditions of the SVD filter can be expected, and it was shown that the conditions of the orthogonality of the signals and the amplitude ratio would be major factors in the performance of the SVD filter.

First assume that $\hat{p}(k) = p_1(k) + p_2(k)$ and two cases are considered and compared. Case 1 is that $p_1(k)$ and $p_2(k)$ are orthogonal and given as $p_1(k) = a_1 \sin(10\pi k \Delta t)$, $p_2(k) = a_2 \sin(100\pi k \Delta t)$. $\Delta t$ is the sampling time and is equal to 0.001 sec. $k$ is between 0 and 200. Case 2 is the condition that $p_1(k)$, $p_2(k)$ are not orthogonal and $p_1(k) = a_1 \sin(10\pi k \Delta t)$, $p_2(k) = a_2 \sin(95\pi k \Delta t)$. $\Delta t$ and $k$ are the same as in Case 1. Taking the SVD on $\hat{p}(k)$, we obtain the singular vectors $(s_1, s_2, s_3, s_4)$, respectively. Reconstruct $\hat{p}_1(k)$, $\hat{p}_2(k)$ by $\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4$, that is

\[
\hat{p}_1(k) = \hat{r}_1 \hat{\sigma} \hat{s}_1^T, \quad (11)
\]
\[
\hat{p}_2(k) = \hat{r}_2 \hat{\sigma} \hat{s}_2^T. \quad (12)
\]

Then taking the SVD on $p_1(k), p_2(k)$ to obtain the corresponding singular vectors $(s_1, s_2)$, $(s_3, s_4)$. If the conditions Eq. (10) hold, then a perfect decomposition can be expected. To evaluate the performance of the SVD, an index is defined by using the square summation of difference between $\hat{p}_i(k)$ and $p_i(k)$, that is

\[
e_s = \sum_{k=0}^{k=200} (p_i(k) - \hat{p}_i(k))^2. \quad (13)
\]

When $e_s$ approaches zero, it means SVD can decompose the $\hat{p}(k)$ into $p_1(k), p_2(k)$ very well. It is shown in Fig. 4 and Fig. 5 that the decomposition is very well as the ratio $a_1/a_2$ is not equal to 1.

![Fig.3. Error $e_s$ for different signal ratio as $p_1(k)$ and $p_2(k)$ are orthogonal](image)

![Fig.4. Comparison between reconstructed signal $\hat{p}_i(k)$ and original signal $p_i(k)$ for $p_1(k), p_2(k)$ are orthogonal and $a_1/a_2$ is not equal to 1.](image)
Fig. 5. Comparison between reconstructed signal $\hat{p}_1(k)$ and original signal $p_1(k)$ for $p_1(k)$, $p_2(k)$ are orthogonal and $a_1/a_2$ is not equal to 1.

However, as $a_1/a_2$ equals 1, the SVD decomposes the $p(k)$ into a linear combination of $p_1(k)$ and $p_2(k)$ with the phase shift as shown in Fig. 6 and Fig. 7.

Fig. 6. Comparison between reconstructed signal $\hat{p}_1(k)$ and original single $p_1(k)$ for $p_1(k)$, $p_2(k)$ are orthogonal and $a_1/a_2$ is equal to 1.

Fig. 7. Comparison between reconstructed signal $\hat{p}_2(k)$ and original signal $p_2(k)$ for $p_1(k)$, $p_2(k)$ are orthogonal and $a_1/a_2$ is equal to 1.

It is shown that the amplitude ratio also plays an important role in the SVD filtering technique. For Case 2, Fig. 8 shows the plot of $e_s$ under different amplitude ratios $a_1/a_2$.

Fig. 8. Error $e_s$ for different signal ratio as $p_1(k)$ and $p_2(k)$ are not orthogonal

It is shown that if $p_1(k)$ and $p_2(k)$ are not orthogonal, the SVD using Eq. (10) does not perform very well unless the ratio $a_1/a_2$ is far away from 1. When compared from Fig. 3 to Fig. 8, it is clear that the higher orthogonality, the easier the SVD can reconstruct the signal using the SVD.

The second condition is the amplitude ratio, when $p_1(k)$ and $p_2(k)$ are not orthogonal, then $a_1/a_2$ or $a_1/a_2 > 1.5$ are the sufficient condition to make Eq. (10) is near true. In Fig. 9 and Fig. 10,
Fig. 9. Comparison between reconstructed signal $\hat{p}_1(k)$ and original signal $p_1(k)$ for $p_1(k)$, $p_2(k)$ are not orthogonal and $a_1/a_2$ is far away from 1.

Fig. 10. Comparison between the reconstructed signal $\hat{p}_1(k)$ and the original signal $p_1(k)$ for $p_1(k)$, $p_2(k)$ are not orthogonal and $a_1/a_2$ is far away from 1.

we have shown Case 2 is more difficult than Case 1 for SVD to make Eq.(10) true, but an acceptable decomposition also can be reached as $a_1/a_2$ are not so close. In Fig. 11 and Fig. 12.

Fig. 11. Comparison between reconstructed signal $\hat{p}_1(k)$ and original signal $p_1(k)$ for $p_1(k)$, $p_2(k)$ are not orthogonal and $a_1/a_2$ is equal to 1.

Fig. 12. Comparison between reconstructed signal $\hat{p}_1(k)$ and original signal $p_1(k)$ for $p_1(k)$, $p_2(k)$ are not orthogonal and $a_1/a_2$ is close to 1, $p_1(k) - \hat{p}_1(k)$ and $p_2(k) - \hat{p}_2(k)$ are total different. Therefore, it is difficult to separate $p_1(k)$ and $p_2(k)$.

THE THEORETICAL DEVIATION FOR OTHERGONIATY AND AMPLITUDE RATIO

To explain the behavior that why amplitude ratio and orthogonality affect the performance of the SVD, theoretical derivation is given in the following:

For simplifying the derivation, here assumes

$p(k) = p_1(k) + p_2(k)$ and $p_1(k) = p_1(k) + p_2(k)$

both are the sinusoidal signals with different frequency and can be decomposed into two singular vectors.

$$H_1(0) = R_1 \Sigma_1 S_1 = r_1 \sigma_1 s_1^T + r_2 \sigma_2 s_2^T,$$

$$H_2(0) = R_2 \Sigma_2 S_2 = r_3 \sigma_3 s_3^T + r_4 \sigma_4 s_4^T,$$

(14)

$H_i(0)$ are formed by the signal $p_i(k)$, and

$$\hat{H}(0) = H_1(0) + H_2(0),$$

(15)

Substituting Eq. (14) into Eq. (15),

$$\hat{H}(0) = r_1 \sigma_1 s_1^T + r_2 \sigma_2 s_2^T + r_3 \sigma_3 s_3^T + r_4 \sigma_4 s_4^T,$$

(16)

In an ideal decomposition, $s_i,i = 1~4$, is a singular vector of $\hat{H}(0)$, namely that the input vector $x = s_i(k)$ and the gain of $\hat{H}(0)$ or $\frac{\|Hx\|}{\|x\|}$ is the extreme
value. Applying the Lagrange multiplier and define the cost function J as

\[ J = x^T \hat{H} \hat{H} x + \lambda (1 - x^T x) \]  

(17)

Let \( \frac{\partial J}{\partial x} = 0 \), then

\[ \hat{H}^T \hat{H} x = \lambda x \]  

(18)

Therefore, Eq. (18) is the sufficient condition for \( x \) to be a singular vector.

Under the assumption that the singular vectors \( s_i \), \( i=1\sim4 \), form an orthonormal set meaning that \( s_i^T s_j = 0 \), for \( i \neq j \), and then substitutes the Eq. (16) into Eq. (18), it becomes

\[ \hat{H}^T \hat{H} x = (\sigma_1^2 s_1 s_1^T + \sigma_2^2 s_2 s_2^T + \sigma_3^2 s_3 s_3^T + \sigma_4^2 s_4 s_4^T) x \]  

(19)

If \( x = s_i \), Eq. (19) becomes

\[ \hat{H}^T \hat{H} s_i = (\sigma_1^2 s_1 s_1^T + \sigma_2^2 s_2 s_2^T + \sigma_3^2 s_3 s_3^T + \sigma_4^2 s_4 s_4^T) s_i \]  

(20) \[ = \sigma_i^2 s_i = \lambda s_i \]  

Therefore, for a compound signal \( \hat{H}(k) \), the condition that SVD can decompose the \( \hat{H}(k) \) well and satisfy Eq. (10) is \( s_i \), \( i=1\sim4 \), are orthonormal set. That is,

\[ s_i^T s_j = 0 \] \( i \neq j \)  

(21)

When \( s_i \), \( i=1\sim4 \), are orthonormal set, then \( s_i \) is happened to be the singular vector of \( \hat{H}(k) \). Therefore, it is very easy to reconstruct the individual original signal by SVD and separate the unwanted signal.

The second condition is the amplitude ratio, because \( s_i \) has been normalized to \( |s_i| = 1 \), therefore, the amplitude ratio that would deteriorate the SVD filter performance is \( \sigma_i / \sigma_j, i \neq j \). Assumed a special case, that is

\[ \sigma_i = \sigma_j \]  

(22)

This would be the worst case that SVD can’t separate the signal from noise well.

Under this condition, the singular vector of \( \hat{H}(0) \) can be a combination of \( (s_i + s_j) / \sqrt{\sigma_i^2 + \sigma_j^2} \), especially when \( s_i(k), s_j(k) \) are not an orthonormal set because

\[ \hat{H}^T \hat{H}(s_i + s_j) / \sqrt{\sigma_i^2 + \sigma_j^2} \]

\[ = (\sigma_i^2 s_i s_i^T + \sigma_j^2 s_j s_j^T + \sigma_i^2 s_i^T s_i + \sigma_j^2 s_j^T s_j)(s_i + s_j) / \sqrt{2\sigma_i} \]

\[ = \sigma_i^2 (s_i + s_j) / \sqrt{2\sigma_i} = \lambda (s_i + s_j) \]

It means, if \( \sigma_i = \sigma_j \), then SVD could not separate the signal to individual one, but a mixed signal would be a singular vectors. Furthermore, as shown in Fig. 8, under the condition of un-orthogonality, \( (s_i + s_j) / \sqrt{\sigma_i^2 + \sigma_j^2} \) is the only choice for SVD to obtain their singular vector, but not \( s_i, s_j \), unless \( \sigma_i \) is away from \( \sigma_j \). This is the reason that SVD could not remove 100% noise from signal. There is always some noise could not be removed perfectly because some singular vectors of signal have the singular values not far away from the singular values of noise.

**SYSTEM IDENTIFICATION FOR HIGH FREQUENCY MODE BY DOPING AN OPTIMUM SIGNAL**

Based on the previous discussion, this section proposes a new concept for system identification. The novel idea is to dope a signal \( d(k) \) into the experimental data \( \hat{p}(k) \) to obtain the high frequency mode information. The signal \( d(k) \) is obtained from a pulse response of second order system, which is given as:

\[ G_d(s) = \frac{a^2}{s^2 + 2\zeta s + \omega^2} \]  

(23)

If the pole of the system to be identified is a real pole, then the \( d(k) \) is obtained from the first order system,

\[ G_d(s) = \frac{a\omega}{s + \omega} \]  

(24)

Doping the signal \( d(k) \) into \( \hat{p}(k) \) will help the SVD algorithm to separate the noise from high frequency model response based on the previous discussion. That is one can improve the identification results by doping.
an artificial signal $d(k)$, and then apply the SVD to determine the information of the system. However, how to determine the parameters of the $G_d(s)$ is a critical issue in this approach. An optimization method integrated with the characteristics of SVD on system identification has been proposed in this section. For illustration purpose, the two degree of freedom (2-DOF) structural system is assumed and $\hat{p}(k)$ is composed of three parts, where $p_i(k)$ is the response of a lower order mode, $p_2(k)$ is the response of a higher frequency mode and $p_s(k)$ is the noise. The amplitude of $p_i(k)$ is larger than $p_2(k)$ and $p_s(k)$ for the structural system because $p_i(k)$ normally corresponds to the first mode. $p_2(k)$ is easily submerged in the noise and it leads to worse identification accuracy for the high frequency modes using SVD because the singular vectors of $p_2(k)$ are close to the singular values of noise. Thus, to save the $p_2(k)$ from $p_s(k)$ or separate $p_2(k)$ and $p_s(k)$ by doping a signal becomes realizable because the doping signal can increase the singular values of $p_2(k)$ and the amplitude ratio can be set away from the singular values of noise. By utilizing the optimization process and ERA, it is possible to find an optimal frequency, damping and amplitude of the doping signal, and to increase the identification accuracy of the higher frequency mode. Because the doped signal is found by optimization technique, we call the doping signal as “an optimum signal”.

To perform the optimization, the cost function $J$ is defined as:

$$J = \sum_{k=0}^{N} (|\hat{p}(k)| - |p_o(k)|)^2,$$

(25)

$\hat{p}(k)$ is the pulse response of given system with noise polluted $p_o(k)$ is the pulse response of identified system

This optimization problem is to find an optimal frequency $\omega$, damping $\zeta$ and amplitude $a$ of optimum signal such that $J$ is minimized. The optimization search process begins from an initial guess $\omega_0, \zeta_0, a_0$, and then calculates the gradient of cost function $J$ in Eq. (34) and iterates the parameters toward the direction of the gradient to one step length. Repeat this process until the tolerance has been satisfied. Finally, the optimal parameters $(\omega, \zeta, a)$ can be obtained.

In calculating the cost function, one can use Eq.(31) to generate the doping signal $d(k)$, and then combine this signal with pulse response $\hat{p}(k)$ of system becomes $\hat{p}_d(k)$. Use $\hat{p}_d(k)$ to form Hankel matrix $\hat{H}_d(0)$ and decompose $\hat{H}_d(0)$ by SVD. Finally, remove the redundant order of $\hat{H}_d(0)$ to obtain $H_d(0)$, that is:

$$\hat{H}_d(0) = r_1 \sigma_1 s_1^T + r_2 \sigma_2 s_2^T + \ldots + r_N \sigma_N s_N^T,$$

(26)

$$H_d(0) = r_1 \sigma_1 s_1^T + r_2 \sigma_2 s_2^T + \ldots + r_N \sigma_N s_N^T,$$

(27)

where $n$ is the order of system.

Deduct the signal $d(k)$ from $H_d(0)$ to obtain $\hat{H}_d(0)$. Finally, one can use $\hat{H}_d(0)$ and Eq. (4) to determine $A_d, B_d, C_d, D_d$ and $p_d(k)$, then the cost function at each step can be determined. After iteration, the optimal doping signal can be obtained.

**NUMERICAL SIMULATION AND DISCUSSION**

To illustrate the optimum signal technique, a fourth order system is established as a working example. The transfer function of the system is given as below:

$$\frac{W(s)}{U(s)} = G(s) = \frac{\omega_0^2}{(s^2 + 2\zeta_1 \omega_0 s + \omega_0^2)(s^2 + 2\zeta_2 \omega_2 s + \omega_2^2)},$$

(28)

where $\zeta_1, \zeta_2$ are the damping ratios of the system, $\omega_1, \omega_2$ are the nature frequencies of the system. $W(s)$ is the Laplace Tranform of the output signal, and $U(s)$ is that of the input signal. The parameters of system in Eq. (28) are $\omega_1 = 5$ rad/sec, $\omega_2 = 50$ rad/sec, $\zeta_1 = 0.5$, $\zeta_2 = 0.5$.

Eq. (28) can be transformed to a state space model and given as:

$$\dot{q} = Aq + Bu,$$

(29)

$$W = Cq + Du,$$

where $A(B,C,D)$ is the system model to be identified.
To evaluate the identification performance, an index is defined by using the average time-domain percentage error for frequency $\omega_2$ and is given as:

$$e_p = \frac{1}{N} \sum_{n=1}^{N} \left| W(n) - W_d(n) \right| / W_{\text{max}},$$  \hspace{1cm} (30)

where $W(n)$ is the output response of system, $W_d(n)$ is the output response obtained from the identified system, $W_{\text{max}}$ are the maximum absolute values of $W(n), e_p$ is the average percentages error.

The optimization process starts from an initial values of $\omega = 60 \text{ rad/sec}, \zeta = 0.1$ and $a=0.01$. The doping signal $d(k)$ is the pulse response of

$$G_d(s) = \frac{a\omega^2}{s^2 + 2\zeta \omega s + \omega^2},$$  \hspace{1cm} (31)

After the iteration process, the optimal parameters are always located near the $\omega_2$ and $\zeta_2$, but the optimal $a$ is varying according to different S/N. The final result of optimization is listed on Table 1.

<table>
<thead>
<tr>
<th>S/N</th>
<th>$\omega_{\text{opt}}$</th>
<th>$\zeta_{\text{opt}}$</th>
<th>$a_{\text{opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>50.298</td>
<td>0.372</td>
<td>2.641</td>
</tr>
<tr>
<td>80</td>
<td>50.127</td>
<td>0.356</td>
<td>2.523</td>
</tr>
<tr>
<td>60</td>
<td>49.766</td>
<td>0.333</td>
<td>6.694</td>
</tr>
<tr>
<td>40</td>
<td>49.249</td>
<td>0.298</td>
<td>10.17</td>
</tr>
<tr>
<td>30</td>
<td>48.848</td>
<td>0.273</td>
<td>16.59</td>
</tr>
<tr>
<td>20</td>
<td>48.272</td>
<td>0.241</td>
<td>41.937</td>
</tr>
<tr>
<td>10</td>
<td>47.369</td>
<td>0.205</td>
<td>1.18</td>
</tr>
<tr>
<td>5</td>
<td>46.432</td>
<td>0.194</td>
<td>2.56</td>
</tr>
</tbody>
</table>

The simulation results show the $e_p$ for different S/N between the optimum signal method and the ERA method are shown in Table 2.

<table>
<thead>
<tr>
<th>S/N</th>
<th>Optimum signal</th>
<th>ERA</th>
<th>Butterworth</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.2229e-12</td>
<td>3.5376e-12</td>
<td>3.5376e-12</td>
</tr>
<tr>
<td>80</td>
<td>8.0380e-12</td>
<td>7.0100e-12</td>
<td>7.0100e-12</td>
</tr>
<tr>
<td>60</td>
<td>2.9577e-11</td>
<td>1.0520e-11</td>
<td>1.0520e-11</td>
</tr>
<tr>
<td>40</td>
<td>2.3016e-10</td>
<td>4.8965e-10</td>
<td>4.8965e-10</td>
</tr>
<tr>
<td>30</td>
<td>1.0431e-9</td>
<td>1.2821e-8</td>
<td>1.2821e-8</td>
</tr>
<tr>
<td>20</td>
<td>8.3678e-9</td>
<td>4.4284e-8</td>
<td>4.4284e-8</td>
</tr>
<tr>
<td>10</td>
<td>4.1654e-8</td>
<td>1.7247e-7</td>
<td>1.7247e-7</td>
</tr>
<tr>
<td>5</td>
<td>6.3748e-7</td>
<td>6.4853e-7</td>
<td>6.4853e-7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>S/N</th>
<th>Optimum signal</th>
<th>ERA</th>
<th>Butterworth</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.8625 + 0.1877i</td>
<td>0.8625 + 0.1877i</td>
<td>0.8625 + 0.1877i</td>
</tr>
<tr>
<td>80</td>
<td>0.8680 + 0.1725i</td>
<td>0.8680 + 0.1725i</td>
<td>0.8680 + 0.1725i</td>
</tr>
<tr>
<td>60</td>
<td>0.8696 + 0.1700i</td>
<td>0.8696 + 0.1700i</td>
<td>0.8696 + 0.1700i</td>
</tr>
<tr>
<td>40</td>
<td>0.8690 + 0.2068i</td>
<td>0.8690 + 0.2068i</td>
<td>0.8690 + 0.2068i</td>
</tr>
<tr>
<td>30</td>
<td>0.8718 + 0.1698i</td>
<td>0.8718 + 0.1698i</td>
<td>0.8718 + 0.1698i</td>
</tr>
<tr>
<td>20</td>
<td>0.8718 - 0.1698i</td>
<td>0.8718 - 0.1698i</td>
<td>0.8718 - 0.1698i</td>
</tr>
<tr>
<td>10</td>
<td>0.8690 - 0.2068i</td>
<td>0.8690 - 0.2068i</td>
<td>0.8690 - 0.2068i</td>
</tr>
<tr>
<td>5</td>
<td>0.8718 + 0.1698i</td>
<td>0.8718 + 0.1698i</td>
<td>0.8718 + 0.1698i</td>
</tr>
</tbody>
</table>

From Table 1, it is demonstrated that when the noise has the same singular value as the high frequency mode, the noise pollution is the most serious, that is the case of S/N equaling 20~40. The conventional approach that filters the signal by using Butterworth filter before performing the ERA identification process is also performed and the results are shown in Table 1 and 2. It is clear that applying such a filter is not helpful in identification of the high frequency mode. The identification accuracy is almost the same as that of the ERA method only. The identified discrete poles locus for S/N=5~1000 has been plotted in Fig.13. Figure 14 and 15 show the comparison of sinusoidal responses by using the optimum signal method and the ERA method at S/N=30.
Fig.13. Identified discrete pole locus using the optimum signal and ERA methods for S/N=10~1000

Fig.14. Sinusoid response for the high frequency mode using ERA method

Fig.15. Sinusoid response at the second mode frequency of ERA using optimum signal method

Again, the results show the optimum signal method can increase the identification accuracy on the high frequency mode significantly.

**CONCLUSION**

This paper has proposed a system identification methodology by using ERA and SVD with a doping signal such that the noise effect can be attenuated and a more accurate identification result of high frequency modes can be obtained. The analysis is performed on the conditions how the SVD can separate signal and noise. One unique feature of this paper is to use examples and theoretical derivation to determine how the singular values ratio and orthogonality affect the identification results by using SVD on identification. Based on the observations, a novel idea is proposed that using a doping signal to extract the information of the high frequency mode. The parameters of the doping signal are obtained by the optimization technique. The simulation results show that the proposed technique can improve the identification accuracy significantly as compared to the SVD alone or the SVD with a Butterworth filter.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>State vector of system</td>
</tr>
<tr>
<td>( U )</td>
<td>Input vector of system</td>
</tr>
<tr>
<td>( W )</td>
<td>Output vector of system</td>
</tr>
<tr>
<td>( \hat{A}, \hat{B}, \hat{C}, \hat{D} )</td>
<td>Discrete time model of system</td>
</tr>
<tr>
<td>( \hat{A}, \hat{B}, \hat{C}, \hat{D} )</td>
<td>Identified discrete time model of system</td>
</tr>
<tr>
<td>( \hat{H}(0), \hat{H}(1) )</td>
<td>Hankel matrix formed by the system response or signal</td>
</tr>
<tr>
<td>( \hat{H}(0) )</td>
<td>Hankel matrix formed by the noise-polluted pulse response</td>
</tr>
<tr>
<td>( \hat{H}_r )</td>
<td>The reduced Hankel matrix formed by the rest ( N-n ) singular value of ( \hat{H}(0) )</td>
</tr>
<tr>
<td>( \hat{R}_N, \hat{\Sigma}_N, \hat{S} )</td>
<td>The items of singular value decomposition of ( \hat{H}(0) )</td>
</tr>
<tr>
<td>( \hat{R}_N, \hat{\Sigma}_N, \hat{S} )</td>
<td>The items of singular value decomposition of ( \hat{H}(0) )</td>
</tr>
<tr>
<td>( \sigma, \hat{\sigma} )</td>
<td>Singular value of ( \hat{H}(0) ) and ( \hat{H}(0) )</td>
</tr>
<tr>
<td>( \hat{\xi}_1, \hat{\xi}_2 )</td>
<td>the damping ratio of the system</td>
</tr>
<tr>
<td>( \omega_1, \omega_2 )</td>
<td>the nature frequency of the system</td>
</tr>
<tr>
<td>( p(k) )</td>
<td>A signal or system response</td>
</tr>
<tr>
<td>( p_i(k) )</td>
<td>i th component of ( p(k) )</td>
</tr>
<tr>
<td>( r, \hat{r} )</td>
<td>i th column of ( R ) and ( \hat{R} )</td>
</tr>
<tr>
<td>( s, \hat{s} )</td>
<td>i th column of ( S ) and ( \hat{S} ) or singular vectors of ( H(0) ) and ( \hat{H}(0) )</td>
</tr>
<tr>
<td>( J )</td>
<td>Cost function</td>
</tr>
<tr>
<td>( e_s )</td>
<td>square summation of difference between ( \hat{s}_i ) and ( s_i )</td>
</tr>
<tr>
<td>( e_o )</td>
<td>average percentage error</td>
</tr>
</tbody>
</table>

**REFERENCE**


掺雜最佳化訊號之奇異值分解於系統識別之應用

蕭庭郎 鄭志希 蔡孟勳
國立中正大學機械工程學系

摘 要

本文提出一種利用摻雜訊號的ERA來做系統識別的方法以降低雜訊的影響。摻雜的訊號是整合最佳化手法與奇異值分解(SVD)而求得。文中提出理論推導來解釋使用最佳化訊號的必要性及奇異值分解與最佳化訊號間的關係。最後,模擬結果顯示本文提出的摻雜最佳化訊號的ERA與沒有摻雜訊號的方法比較,可以有效的提高系統識別精度。